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Approximate Radiative Solutions of the Yang-Mills Field Equations

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Abstract

In this paper we look for the asymptotic radiative solutions of the Yang-Mills field equations. Considering the potential of the Yang-Mills field as a connection in a principal fibre bundle gives us a fully covariant formalism similar to the formalism of the General Relativity. Then we apply directly the results obtained by Mme Choquet-Bruhat for the gravitational field by means of the W.K.B. method. After deriving the equations for the asymptotic waves and interpreting the zero-order conditions as the initial conditions, we consider some known trivial solutions of the Yang-Mills field equations as the 'background field' and construct the asymptotic waves explicitly. All the solutions considered turn out to be of the electromagnetic type, with some extra restrictions of the algebraic type.

1. *Introduction*

This paper is basically a generalisation of the work done by Mme Choquet-Bruhat (1964) on the approximate radiative solutions of Einstein's equations. By using the fact that the Yang-Mills field theory can be constructed in a purely geometrical way as a special case of the Riemannian fibre bundle (Kerner, 1968), we can apply directly almost all the results obtained by Mme Choquet-Bruhat for the gravitational field to the case of the Yang-Mills field.

The approximate radiative solutions of Einstein's equations are constructed by using the W.K.B. method. This means that given a system of partial differential equations,

$$L_A(f_B) = 0 \tag{1.1}$$

with $A, B, \ldots = 1, 2, \ldots, N$, on the n-dimensional differentiable manifold V_n, $x \in V_n$, we search for the solutions f_B of the form

$$f_B(x, \omega\varphi) = \sum_{p=0}^{\infty} \omega^{-p} \overset{p}{f}_B(x, \omega\varphi) \tag{1.2}$$

Here φ is a scalar function on V_n called the phase, and ω a parameter which is chosen to be great enough to ensure the asymptotic validity of the series (1.2).

Then equation (1.1) can be expanded into a formal series

$$L_A(f_B) = \sum_{p=-m}^{\infty} \omega^{-p} \overset{p}{F}_A(x, \omega\varphi) \tag{1.3}$$

As has been pointed out by Garding *et al.* (1964), we note that the formal series (1.2) is an asymptotic wave for the system (1.1) if

$$\overset{q}{F}_A(x, \xi) = 0 \tag{1.4}$$

for any x and ξ, ξ being a real parameter.

The finite sum

$$f_A(x, \omega\varphi) = \sum_{p=0}^{r} \omega^{-p} \overset{p}{f}_A(x, \omega\varphi) \tag{1.5}$$

will be called an approximate wave of the order r in the domain $\Omega \subset V_n$ if for any $x \in \Omega$ and any value of φ

$$|L_A(f_B)| \leqslant M\omega^{-r} \tag{1.6}$$

for some suitably chosen constant M.

Applying this method to the Einstein equations *in vacuo*, by expanding the metric tensor into series

$$g_{ij}(x, \omega\varphi) = \overset{0}{g}_{ij}(x) + \frac{1}{\omega}\overset{1}{g}_{ij}(x, \omega\varphi) + \frac{1}{\omega^2}\overset{2}{g}_{ij}(x, \omega\varphi) + \cdots \tag{1.7}$$

with $i, j, \ldots = 0, 1, 2, 3$, and then expanding the Ricci tensor as

$$R_{ij} = \omega \overset{-1}{R}_{ij} + \overset{0}{R}_{ij} + \frac{1}{\omega}\overset{1}{R}_{ij} + \cdots \tag{1.8}$$

one obtains a lot of useful information about the approximate waves of order 0, i.e. for the case $\overset{-1}{R}_{ij} = 0$, R_{ij} finite; of order 1, i.e. for the case $\overset{-1}{R}_{ij} = 0$ and $\overset{0}{R}_{ij} = 0$, R_{ij} finite, and so on for higher orders.

Now, the same method can be applied to the Yang-Mills field equations. We recall here that these field equations can be derived exactly in the same way as the Einstein equations by constructing a special metric over the principal fibre bundle $P(V_4, G)$. Here the base of the fibre bundle V_4 is any four-dimensional Riemannian manifold with the usual space-time structure and the metric g_{ij}; G is a semi-simple and compact Lie group, called the gauge group. G is isomorphic to the fibre in $P(V_4, G)$. Next we introduce the connection form in $P(V_4, G)$, which in local coordinates can be written as $A_\alpha{}^a(x)$, with $a, b, \ldots = 1, 2, \ldots, K$, $K = \dim G$, and $\alpha, \beta, \ldots = 1, 2, \ldots, K + 4$. We can always choose such a coordinate system in any neighbourhood in $P(V_4, G)$, in which $A_b{}^a = \delta_b{}^a$, and the only non-trivial components of $A_\alpha{}^a$ are $A_j{}^a(x)$. The corresponding curvature form can be written then in local coordinates as

$$F^a_{ij} = \partial_i A_j{}^a - \partial_j A_i{}^a + C^a_{bc} A_i{}^b A_j{}^c \tag{1.9}$$

where C^a_{bc} are the structure constants of the group G. We also introduce an invariant metric structure on G by putting

$$g_{ab} = C^c_{ad} C^d_{cb} \tag{1.10}$$

Finally, we construct the following metric on $P(V_4, G)$ from g_{ij}, g_{ab} and $A_j{}^a$:

$$\gamma_{\alpha\beta} = \left(\begin{array}{c|c} g_{ij} + g_{ab} A_i{}^a A_j{}^b & g_{ab} A_j{}^b \\ \hline g_{ab} A_i{}^a & g_{ab} \end{array}\right) \tag{1.11a}$$

whose inverse is

$$\gamma^{\alpha\beta} = \left(\begin{array}{c|c} g^{ij} & -g^{ij} A_j{}^b \\ \hline -g^{ij} A_i{}^a & g^{ab} + g^{ij} A_i{}^a A_j{}^b \end{array}\right) \tag{1.11b}$$

$\alpha, \beta = 1, 2, \ldots, K+4$.

We now construct the curvature scalar R for our $K+4$ dimensional fibre bundle from $\gamma_{\alpha\beta}$ and use it as the Lagrangian density for the variational principle, to obtain the Yang-Mills field equations from

$$(R_{\alpha\beta} - \tfrac{1}{2}\gamma_{\alpha\beta} R)\,\delta\gamma^{\alpha\beta} = 0 \tag{1.12}$$

$R_{\alpha\beta}$ being the Ricci tensor corresponding to the metric $\gamma_{\alpha\beta}$.

The variations $\delta\gamma^{\alpha\beta}$ are not completely arbitrary, because the metric (1.11) has to preserve its particular form, so that we are varying only the g^{ij} and $A_i{}^a$, and (1.12) gives us indeed two systems of equations: one of the form

$$A_k{}^a R_{ab} - R_{bk} = 0 \tag{1.13a}$$

corresponding to the variations of the $A_i{}^a$; the other, corresponding to the variations of the g^{ij}, is of the form

$$R_{jk} - \tfrac{1}{2} g_{jk} R - A_j{}^a R_{ak} = 0 \tag{1.13b}$$

and can be regarded as a definition of the energy-momentum tensor for the Yang-Mills field.

We should also remark that in our case the Riemannian scalar $R = \gamma^{\alpha\beta} R_{\alpha\beta}$ is not null in general.

In the case of the Minkowskian space-time, which we shall investigate from now on, the variations δg^{ij} vanish identically and we are left with one system of equations only, viz.

$$A_k{}^a R_{ab} - R_{bk} = 0 \tag{1.13a}$$

which can also be written explicitly as

$$\partial^i F^a_{ij} + C^a_{bc} A^{bi} F^c_{ij} = 0 \tag{1.14}$$

2. *Waves of Zeroth Order*

Accordingly to what was stated above, we shall call waves of order 0 the solutions of the equations

$$\overset{0}{A}{}_k{}^a \overset{-1}{R}_{ab} - \overset{-1}{R}_{bk} = 0 \tag{2.1}$$

Developing the potential $A_j{}^a(x, \omega\varphi)$ into a series of the form (1.2) and keeping only terms up to the second order in ω^{-1}, i.e. assuming the potential to be

$$A_j{}^a = \overset{0}{A}{}_j{}^a(x) + \frac{1}{\omega}\overset{1}{A}{}_j{}^a(x, \omega\varphi) + \frac{1}{\omega^2}\overset{2}{A}{}_j{}^a(x, \omega\varphi) \tag{2.2}$$

and then putting this expression into the formulae (1.11a) and (1.11b), we obtain the following matrices for $\overset{p}{\gamma}_{\alpha\beta}$:

$$\overset{0}{\gamma}_{\alpha\beta} = \left(\begin{array}{c|c} g_{ij} + g_{ab}\overset{0}{A}{}_i{}^a \overset{0}{A}{}_j{}^a & g_{ab}\overset{0}{A}{}_j{}^b \\ \hline g_{ab}\overset{0}{A}{}_i{}^a & g_{ab} \end{array}\right)$$

$$\overset{1}{\gamma}_{\alpha\beta} = \left(\begin{array}{c|c} g_{ab}(\overset{0}{A}{}_i{}^a \overset{1}{A}{}_j{}^b + \overset{1}{A}{}_i{}^a \overset{0}{A}{}_b{}^a) & g_{ab}\overset{1}{A}{}_j{}^b \\ \hline g_{ab}\overset{1}{A}{}_i{}^a & 0 \end{array}\right)$$

$$\overset{2}{\gamma}_{\alpha\beta} = \left(\begin{array}{c|c} g_{ab}(\overset{0}{A}{}_i{}^a \overset{2}{A}{}_j{}^b + \overset{2}{A}{}_i{}^a \overset{0}{A}{}_j{}^b + 2\overset{1}{A}{}_i{}^a \overset{1}{A}{}_j{}^b) & g_{ab}\overset{2}{A}{}_j{}^b \\ \hline g_{ab}\overset{2}{A}{}_i{}^a & 0 \end{array}\right) \tag{2.3a}$$

and

$$\overset{0}{\gamma}{}^{\alpha\beta} = \left(\begin{array}{c|c} g^{ij} & -g^{ij}\overset{1}{A}{}_j{}^b \\ \hline -g^{ij}\overset{0}{A}{}_j{}^a & g^{ab} + g^{ij}\overset{0}{A}{}_i{}^a \overset{0}{A}{}_j{}^b \end{array}\right)$$

$$\overset{1}{\gamma}{}^{\alpha\beta} = \left(\begin{array}{c|c} 0 & -g^{ij}\overset{1}{A}{}_j{}^b \\ \hline -g^{ij}\overset{1}{A}{}_i{}^a & g^{ij}(\overset{0}{A}{}_i{}^a \overset{1}{A}{}_j{}^b + \overset{1}{A}{}_i{}^a \overset{0}{A}{}_j{}^b) \end{array}\right)$$

$$\overset{2}{\gamma}{}^{\alpha\beta} = \left(\begin{array}{c|c} 0 & -g^{ij}\overset{2}{A}{}_j{}^b \\ \hline -g^{ij}\overset{2}{A}{}_i{}^a & g^{ij}(\overset{0}{A}{}_i{}^a \overset{2}{A}{}_j{}^b + \overset{2}{A}{}_i{}^a \overset{0}{A}{}_j{}^b + 2\overset{1}{A}{}_i{}^a \overset{1}{A}{}_j{}^b) \end{array}\right) \tag{2.3b}$$

$\overset{-1}{R}_{\alpha\beta}$ is calculated to be

$$\overset{-1}{R}_{\alpha\beta} = \tfrac{1}{2}\overset{0}{\gamma}{}^{\gamma\delta}\{(\overset{1}{g}{}''_{\alpha\delta} n_\beta + \overset{1}{\gamma}{}''_{\beta\delta} n_\alpha) n_\gamma - \overset{1}{\gamma}{}''_{\gamma\delta} n_\alpha n_\beta - \overset{1}{\gamma}{}''_{\alpha\beta} n_\gamma n_\delta\} \tag{2.4}$$

where

$$n_\alpha = \partial_\alpha \varphi \qquad \text{and} \qquad \overset{1}{\gamma}{}''_{\alpha\delta} = \left.\frac{\partial^2 \overset{1}{\gamma}_{\alpha\delta}(x, \xi)}{\partial \xi^2}\right|_{\xi = \omega\varphi} \tag{2.5}$$

The compactness of the gauge group leads us to the most natural assumption $\partial_a \varphi = n_a = 0$, i.e. $n_\alpha = 0$ for $\alpha = 1, \ldots, K$. Then one easily sees that the components $\overset{-1}{R}_{ab}$ of $\overset{-1}{R}_{\alpha\beta}$ vanish identically, so that finally we have to solve the system

$$\overset{-1}{R}_{ak} = 0 \tag{2.6}$$

The case when $\overset{0}{\gamma}{}^{\alpha\beta} n_\alpha n_\beta \neq 0$ gives immediately the solution

$$\overset{1}{\gamma}{}''_{\alpha\beta} = \theta_\alpha n_\beta + \theta_\beta n_\alpha \tag{2.7}$$

that is

$$\overset{1}{\gamma}{}''_{ab} = 0, \qquad \overset{1}{\gamma}{}''_{aj} = g_{ab} \overset{1}{A}{}''^{b}_{j} = \theta_a(x, \omega\varphi) n_j \tag{2.7a}$$

and

$$\overset{1}{\gamma}{}''_{ij} = g_{ab}(\overset{0}{A}{}_i^a \overset{1}{A}{}''^b_j + \overset{0}{A}{}_j^b \overset{1}{A}{}''^a_i) = \theta_i n_j + \theta_j n_i \tag{2.7b}$$

where θ_α is an arbitrary vector field on $P(M_4, G)$.

The non-trivial case occurs, however, when $\overset{0}{\gamma}{}^{\alpha\beta} n_\alpha n_\beta = 0$, which is equivalent to $g^{ij} n_i n_j = 0$ because of $n_a = 0$.

Then we have

$$\overset{-1}{R}_{\alpha\beta} = \tfrac{1}{2}\overset{0}{\gamma}{}^{\gamma\delta}(\overset{1}{\gamma}{}''_{\alpha\delta} n_\beta n_\gamma + \overset{1}{\gamma}{}''_{\beta\delta} n_\alpha n_\gamma - \overset{1}{\gamma}{}''_{\gamma\delta} n_\alpha n_\beta) = 0 \tag{2.8}$$

which in turn can be written as

$$\overset{-1}{R}_{\alpha\beta} = -\tfrac{1}{2}(n_\alpha \phi''_\beta + n_\beta \phi''_\alpha) = 0 \tag{2.9}$$

where

$$\phi_\alpha = \tfrac{1}{2}\overset{0}{\gamma}{}^{\gamma\delta}(\overset{1}{\gamma}_{\gamma\delta} n_\alpha - \overset{1}{\gamma}_{\alpha\delta} n_\gamma) \tag{2.10}$$

Equations (2.9) are equivalent to

$$\phi''_\alpha = 0 \tag{2.11}$$

When taken into account with the conditions of regularity on $\gamma_{\alpha\beta}$, i.e. the requirement that $\gamma_{\alpha\beta}$ should be always finite, this gives

$$\phi_\alpha = 0 \tag{2.12}$$

One easily verifies that the components ϕ_i vanish identically, and we are left with the following equation only:

$$\begin{aligned}\phi_a &= \tfrac{1}{2} g^{ij}(\overset{1}{\gamma}_{ij} n_a - \overset{1}{\gamma}_{aj} n_i) - \tfrac{1}{2} g^{ij} \overset{0}{A}{}_j^b(\overset{1}{\gamma}_{ib} n_a - \overset{1}{\gamma}_{ab} n_i) \\ &\quad + \tfrac{1}{2}(g^{cb} g^{ij} \overset{0}{A}{}_j^c \overset{0}{A}{}_j^b)(\overset{1}{\gamma}_{cb} n_a - \overset{1}{\gamma}_{ab} n_c) \\ &= -\tfrac{1}{2} g^{ij} \overset{1}{\gamma}_{aj} n_i = -\overset{1}{A}{}_j^b(x, \omega\varphi) g_{ab} n^j \end{aligned} \tag{2.13}$$

from which follows

$$\overset{1}{A}{}_j^a(x, \omega\varphi) n^j = 0 \tag{2.13a}$$

As the characteristics of $n_i = \partial_i \varphi$ are the null vectors with respect to the metric g^{ij}, condition (2.13) means that in the first approximation the first-order correction to the potential, i.e. the asymptotic wave of order 0,

$\overset{1}{A}{}_j{}^b(x,\omega\varphi)$ must be orthogonal to the null-vector n_i, giving the direction of the propagation of the asymptotic wave.

We also note that n_i is a characteristic vector for both the first and the second approximation, because in every case

$$\overset{1}{\gamma}{}^{ij} = 0 \qquad \text{and} \qquad \overset{2}{\gamma}{}^{ij} = 0 \tag{2.14}$$

3. *Radiative Coordinates and Gauge*

It is always possible to make locally the following choice of coordinates:

$$x^0 = \varphi(x), \qquad x^\alpha \text{ being the same for } \alpha \neq 0. \tag{3.1}$$

In these coordinates we have the following obvious relations:

$$n_0 = 1, \qquad n_\alpha = 0 \qquad \text{for } \alpha \neq 0$$

also

$$\overset{0}{\gamma}{}^{00} = 0, \qquad n^0 = 0 \qquad \text{and} \qquad n^\alpha = \overset{0}{\gamma}{}^{\alpha 0} \qquad \text{for } \alpha \neq 0 \tag{3.2}$$

We see then that in these coordinates equation (2.6) takes on a particularly simple form:

$$\overset{-1}{R}_{\alpha\beta} \equiv 0 \qquad \text{for } \alpha, \beta \neq 0 \tag{3.3a}$$

$$\overset{-1}{R}_{0\alpha} \equiv \tfrac{1}{2} n^\beta \overset{1}{\gamma}{}''_{\alpha\beta} = 0 \qquad \text{for } \alpha \neq 0 \tag{3.3b}$$

and

$$\overset{-1}{R}_{00} \equiv -\frac{1}{2}\overset{0}{\gamma}{}^{\alpha\beta} \overset{1}{\gamma}{}''_{\alpha\beta} = 0 \tag{3.3c}$$

Now we remark upon the following property of the metric (1.11a), (1.11b): if we write down the harmonicity conditions

$$F^\alpha = 0 \tag{3.4}$$

where

$$F^\alpha = \frac{1}{\sqrt{-\gamma}} \partial_\beta(\sqrt{(-\gamma)}\, \gamma^{\alpha\beta}) \tag{3.5}$$

then, recalling that in our case $|\gamma| = |\det \gamma_{\alpha\beta}| = |\det g_{ij}| = 1$, we have just

$$F^\alpha = \partial_\beta \gamma^{\alpha\beta} \tag{3.6}$$

Calculating this expression explicitly gives us

$$\partial_\beta \gamma^{\alpha\beta} = \partial_i \gamma^{\alpha i} + \partial_b \gamma^{\alpha b} \tag{3.7}$$

Now we choose the geodesic coordinates in the group manifold G, so that

$$\partial_a A_i{}^b + C^b_{ac} A_i{}^c = 0 \tag{3.8}$$

(The structure constants C^a_{bc} play the role of the connection coefficients in G.) Having this in mind we can proceed further and obtain for $\alpha = j$

$$\partial_i g^{ij} + \partial_b \gamma^{bj} = -g^{ij} \partial_b\ A_i{}^b = g^{ij} C^b_{bc} A_i{}^c \equiv 0 \tag{3.9}$$

because $C^b_{bc} = 0$ for any semi-simple and compact Lie group. For $\alpha = a$ we get

$$\begin{aligned}\partial_\beta \gamma^{\alpha\beta} &= \partial_i \gamma^{ai} + \partial_b(g^{ab} + g^{ij} A_i{}^a A_j{}^b) \\ &= -g^{ij} \partial_i A_j{}^a + g^{ij} C^b_{bc} A_i{}^a A_j{}^c + g^{ij} C^a_{bc} A_i{}^b A_j{}^c \\ &= -\partial^j A_j{}^a \end{aligned} \tag{3.10}$$

So the harmonicity conditions for the metric $\gamma_{\alpha\beta}$ are equivalent to the requirement of the radiative (Lorentz) gauge for the potential $A_j{}^a$, i.e.

$$\partial^j A_j{}^a = 0 \tag{3.11}$$

Further, expanding the harmonicity conditions for the metric (2.3a), (2.3b) into a series in the powers of ω^{-1}, one can easily see that

$$F^\alpha = \overset{0}{F}{}^\alpha + \frac{1}{\omega} \overset{1}{F}{}^\alpha + \cdots \tag{3.12}$$

where

$$\overset{0}{F} = \overset{0}{F}{}^\alpha(\gamma_{\beta\gamma}) + \phi^\alpha \tag{3.13}$$

The fact that $\overset{-1}{R}_{bk} = 0$ implies $\phi^\alpha = 0$ means then that in the zeroth approximation the gauge condition $\partial^j A_j{}^a = 0$ will be conserved. It is also obvious that the Lorentz gauge is the only one having this property.

Radiative coordinates are particularly useful in simplifying the equations for the first-order asymptotic wave, which we shall see in the next section.

4. *Equations for the Waves of the First Order*

Developing equations (1.13a) in the powers of ω, we see that the only terms with the zeroth power are

$$\overset{0}{A}_k{}^a \overset{0}{R}_{ab} - R_{bk} + \overset{1}{A}_k{}^a \overset{-1}{R}_{ab} = 0 \tag{4.1}$$

We also have to satisfy the conditions for the zeroth order; therefore we are left with the following system:

$$\overset{-1}{R}_{bk} = 0 \qquad \text{and} \qquad \overset{0}{A}_k{}^a \overset{0}{R}_{ab} - \overset{0}{R}_{bk} = 0 \tag{4.2}$$

because, as we have seen before, $\overset{-1}{R}_{ab}$ vanish identically.

Direct calculation gives us the following expression for $\overset{0}{R}_{\alpha\beta}$:

$$\overset{0}{R}_{\alpha\beta} = \overset{1}{\Gamma}{}^\delta_{\alpha\beta} n_\delta - \overset{1}{\Gamma}{}^\delta_{\beta\delta} n_\alpha + \partial_\delta \overset{0}{\Gamma}{}^\delta_{\alpha\beta} - \partial_\alpha \overset{0}{\Gamma}{}^\delta_{\beta\delta} + \overset{0}{\Gamma}{}^\delta_{\alpha\beta} \overset{0}{\Gamma}{}^\gamma_{\delta\gamma} - \overset{0}{\Gamma}{}^\gamma_{\alpha\delta} \overset{0}{\Gamma}{}^\delta_{\beta\gamma} \tag{4.3}$$

Recalling that $n^\alpha n_\alpha = 0$, we can write explicitly (all barred quantities correspond to the non-perturbed metric $\overset{0}{\gamma}_{\alpha\beta}$):

$$\overset{0}{R}_{\alpha\beta} = \tfrac{1}{2} \overset{0}{\gamma}{}^{\gamma\delta}(\overset{2}{\gamma}{}''_{\alpha\delta} n_\beta n_\gamma + \overset{2}{\gamma}{}''_{\beta\delta} n_\alpha n_\gamma - \overset{2}{\gamma}{}''_{\gamma\delta} n_\alpha n_\beta) + \mathscr{H}_{\alpha\beta} \tag{4.4}$$

where

$$\mathscr{H}_{\alpha\beta} = I_{\alpha\beta} + II_{\alpha\beta} + III_{\alpha\beta} + IV_{\alpha\beta} + V_{\alpha\beta} \tag{4.5}$$

with

$$I_{\alpha\beta} = \tfrac{1}{2}\overset{1}{\gamma}{}^{\gamma\delta}(\overset{1}{\gamma}{}''_{\alpha\delta} n_\beta n_\gamma + \overset{1}{\gamma}{}''_{\beta\delta} n_\alpha n_\gamma - \overset{1}{\gamma}{}''_{\gamma\delta} n_\alpha n_\beta) \tag{4.6a}$$

$$\begin{aligned} I_{\alpha\beta} = {} & -n^\delta \partial_\delta \overset{1}{\gamma}{}'_{\alpha\beta} + \tfrac{1}{2}\overset{0}{\gamma}{}^{\gamma\delta}(\partial_\gamma \overset{1}{\gamma}{}'_{\alpha\delta} n_\beta + \partial_\gamma \overset{1}{\gamma}{}'_{\beta\delta} n_\alpha) \\ & - \tfrac{1}{2}\overset{0}{\gamma}{}^{\gamma\delta}(\partial_\alpha \overset{1}{\gamma}{}'_{\gamma\delta} n_\beta + \partial_\beta \overset{1}{\gamma}{}'_{\gamma\delta} n_\alpha) + \tfrac{1}{2} n^\delta (\partial_\alpha \overset{1}{\gamma}{}'_{\delta\beta} + \partial_\beta \overset{1}{\gamma}{}'_{\delta\alpha}) \end{aligned} \tag{4.6b}$$

$$\begin{aligned} III_{\alpha\beta} = {} & \tfrac{1}{2}\overset{1}{\gamma}{}'^{\gamma\delta}\{(\overset{1}{\gamma}{}'_{\alpha\delta} n_\beta + \overset{1}{\gamma}{}'_{\beta\delta} n_\alpha) n_\gamma - \overset{1}{\gamma}{}'_{\gamma\delta} n_\alpha n_\beta - \overset{1}{\gamma}{}'_{\alpha\beta} n_\gamma n_\delta)\} \\ & + \tfrac{1}{4} n^\gamma \overset{0}{\gamma}{}^{\delta\eta} \overset{1}{\gamma}{}'_{\delta\eta}(\overset{1}{\gamma}{}'_{\beta\gamma} n_\alpha + \overset{1}{\gamma}{}'_{\alpha\gamma} n_\beta) - \tfrac{1}{4}\overset{0}{\gamma}{}^{\gamma\delta} \overset{0}{\gamma}{}^{\eta\zeta}(\overset{1}{\gamma}{}'_{\eta\varrho} n_\alpha + \overset{1}{\gamma}{}'_{\alpha\delta} n_\eta - \overset{1}{\gamma}{}'_{\alpha\eta} n_\delta) \\ & \times (\overset{1}{\gamma}{}'_{\gamma\zeta} n_\beta + \overset{1}{\gamma}{}'_{\beta\zeta} n_\gamma - \overset{1}{\gamma}{}'_{\beta\gamma} n_\zeta) \end{aligned} \tag{4.6c}$$

$$\begin{aligned} IV_{\alpha\beta} = {} & \overset{1}{\gamma}{}'_{\gamma\delta}\{\tfrac{1}{2}\overset{0}{\gamma}{}^{\delta\eta}(\bar{\Gamma}^\gamma_{\alpha\eta} n_\beta + \bar{\Gamma}^\gamma_{\beta\eta} n_\alpha) - n^\gamma \bar{\Gamma}^\delta_{\alpha\beta}\} \\ & - \tfrac{1}{2}\overset{0}{\gamma}{}^{\gamma\delta} \overset{1}{\gamma}{}'_{\gamma\delta} \overline{\nabla}_\alpha n_\beta + \tfrac{1}{2}\overset{1}{\gamma}{}'_{\beta\delta}(\overset{0}{\gamma}{}^{\gamma\delta} \nabla_\gamma n_\alpha + n^\gamma \bar{\Gamma}^\delta_{\alpha\gamma} - \overset{0}{\gamma}{}^{\eta\zeta} \bar{\Gamma}^\delta_{\eta\zeta} n_\alpha) \\ & + \tfrac{1}{2}\overset{1}{\gamma}{}'_{\alpha\delta}(\overset{0}{\gamma}{}^{\gamma\delta} \overline{\nabla}_\gamma n_\beta + n^\gamma \bar{\Gamma}^\delta_{\beta\gamma} - \overset{0}{\gamma}{}^{\eta\zeta} \bar{\Gamma}^\delta_{\eta\zeta} n_\beta) - \tfrac{1}{2}\gamma'_{\alpha\beta} \overline{\nabla}_\delta n^\delta \end{aligned} \tag{4.6d}$$

$$V_{\alpha\beta} = \bar{R}_{\alpha\beta} \tag{4.6e}$$

The last term will fall out from the equations, because the non-perturbed Ricci tensor verifies equation (1.13a). Equations (4.2) are linear in $\overset{2}{\gamma}{}''_{\alpha\beta}$; but non-homogeneous. The homogeneous term, which is identical to the left-hand side of (2.1), except for the $\overset{1}{\gamma}{}''_{\alpha\beta}$ being replaced by $\overset{2}{\gamma}{}''_{\alpha\beta}$, is not independent; therefore, in order to be satisfied, equations (4.2) reduce themselves to

$$\overset{0}{A}{}_k{}^a G_{ab} - G_{bk} = 0 \tag{4.7}$$

where $G_{\alpha\beta} = I_{\alpha\beta} + II_{\alpha\beta} + III_{\alpha\beta} + IV_{\alpha\beta}$. Moreover, as can be easily seen, these equations do not contain the $\overset{1}{\gamma}{}''_{\alpha\beta}$ and are therefore of the first order.

Before proceeding further we shall give the explicit expressions for the non-perturbed connection coefficients of our metric:

$$\bar{\Gamma}^a_{bc} = 0 \qquad \bar{\Gamma}^j_{bc} = 0 \qquad \bar{\Gamma}^i_{bj} = g^{ik} g_{ab} \overset{0}{F}{}^a_{jk}$$

$$\bar{\Gamma}^a_{bj} = g^{ik} g_{bc} \overset{0}{A}{}_k{}^a \overset{0}{F}{}^c_{ij} + C^a_{cb} \overset{0}{A}{}_j{}^c$$

$$\bar{\Gamma}^i_{jk} = \tfrac{1}{2}(\partial_j \overset{0}{A}{}_k{}^a + \partial_k \overset{0}{A}{}_j{}^a) - \{{}^m_{jk}\} \overset{0}{A}{}_m{}^a + g^{im} g_{bc}(\overset{0}{A}{}_m{}^a \overset{0}{A}{}_k{}^b \overset{0}{F}{}^c_{ij} + \overset{0}{A}{}_m{}^a \overset{0}{A}{}_j{}^b \overset{0}{F}{}^c_{ik})$$

$$\bar{\Gamma}^i_{jk} = \{{}^i_{jk}\} + g^{im} g_{bc}(\overset{0}{A}{}_k{}^b \overset{0}{F}{}^c_{jm} + \overset{0}{A}{}_j{}^b \overset{0}{F}{}^c_{km}) \tag{4.8}$$

where $\{{}^i_{jk}\}$ are the Christoffel symbols corresponding to the space-time metric g_{ij}; they are of course null in the case of the Minkowskian space-time which we are now investigating.

The equations obtained after putting (4.8) into (4.2) are quite cumbersome, therefore we shall pass to the radiative coordinates. In these coordinates the expression for the $\overset{0}{R}_{\alpha\beta}$ is simplified to

$$\overset{0}{R}_{\alpha\beta} = -n^\gamma \partial_\gamma \overset{1}{\gamma}{}'_{\alpha\beta} + \tfrac{1}{2} n^\gamma (\partial_\alpha \overset{1}{\gamma}{}'_{\gamma\beta} + \partial_\beta \overset{1}{\gamma}{}'_{\gamma\alpha}) - \tfrac{1}{2} \overset{1}{\gamma}{}'^{00} \overset{1}{\gamma}{}'_{\alpha\beta} + \tfrac{1}{4} n^\gamma n^\delta \overset{1}{\gamma}{}'_{\beta\gamma} \overset{1}{\gamma}{}'_{\alpha\delta}$$
$$- \tfrac{1}{2} \overset{1}{\gamma}{}'_{\alpha\beta} \overline{\nabla}_\gamma n^\gamma - n^\gamma \overline{\Gamma}{}^\delta_{\alpha\beta} \overset{1}{\gamma}{}'_{\gamma\delta} + \tfrac{1}{2} \overline{\Gamma}{}^0_{\alpha\beta} \overset{0}{\gamma}{}^{\gamma\delta} \overset{1}{\gamma}{}'_{\delta\gamma} - \tfrac{1}{2} \overline{\Gamma}{}^0_{\alpha 0} n^\gamma \overset{1}{\gamma}{}'_{\beta\gamma}$$
$$- \tfrac{1}{2} \overline{\Gamma}{}^0_{\alpha\delta} \overset{0}{\gamma}{}^{\delta\gamma} \overset{1}{\gamma}{}'_{\beta\gamma} + \tfrac{1}{2} \overline{\Gamma}{}^\delta_{\alpha\gamma} \overset{0}{\gamma}{}^{\gamma 0} \overset{1}{\gamma}{}'_{\delta\beta} + \overline{R}_{\alpha\beta} \tag{4.9}$$

(We use here the indices α, β, but recall that in the radiative coordinates (4.2) is a non-trivial equation only for $\alpha, \beta \neq 0$, $\overset{0}{R}_{00}$ being null identically.) Using the conditions of zeroth order we can simplify expression (4.9) even further:

$$\overset{0}{R}{}^{\perp}_{\beta\alpha} \overset{\text{df}}{=} -n^\gamma \overline{\nabla}_\gamma \overset{1}{\gamma}{}'_{\alpha\beta} - \tfrac{1}{2} \overset{1}{\gamma}{}'_{\alpha\beta} \overline{\nabla}_\gamma n^\gamma + \overline{R}_{\alpha\beta} \tag{4.10}$$

where

$$\overline{\nabla}_\gamma \overset{1}{\gamma}{}'_{\alpha\beta} = \partial_\gamma \overset{1}{\gamma}{}'_{\alpha\beta} - \overline{\Gamma}{}^\delta_{\gamma\alpha} \overset{1}{\gamma}{}'_{\beta\delta} - \overline{\Gamma}{}^\delta_{\gamma\beta} \overset{0}{\gamma}{}'_{\alpha\delta} \tag{4.11}$$

Note that in (4.11) we sum only over $\alpha, \beta \neq 0$, excluding any case when an index of any connexion coefficient takes on the value 0.

Putting in the explicit form of the connection coefficients and the components of the metric yields, after the straightforward calculus, the following system:

$$n^i \partial_i \overset{1}{A}{}^{b'}_k - n^i g^{jl} [g_{ac} \overset{0}{F}{}^b_{il} \overset{0}{A}_k{}^a + g_{ad} \overset{0}{F}{}^d_{il} \overset{0}{A}_k{}^a \delta_c{}^b] \overset{1}{A}{}^{c'}_j + (\partial_i n^i) \overset{1}{A}{}^{b'}_k = 0 \tag{4.12}$$

(we remind once more that (4.12) is valid only in the radiative coordinates, the general expression being more complex).

Now, the important result is that in spite of high non-linearity of the original system we have finally obtained a linear system for $\overset{1}{A}{}^{a'}_j$, which can be also written symbolically as

$$n^i \partial_i \overset{1}{A}{}^{b'}_k + \sqcup^{bj}_{ck} \overset{1}{A}{}^{c'}_j = 0 \tag{4.13}$$

where the matrix $\sqcup^{bj}_{ck}$ is given by

$$\sqcup^{bj}_{ck} = -n^i g^{jl} [g_{ac} \overset{0}{F}{}^b_{il} \overset{0}{A}_k{}^a + g_{ad} \overset{0}{F}{}^d_{il} \overset{0}{A}_k{}^a \partial_c{}^b] + (\partial_i n^i) \delta_c{}^b \delta_k{}^j \tag{4.14}$$

Moreover, as one can easily verify, the system (4.13) is always homogeneous, even in non-radiative coordinates. This is the first serious difference between the Yang-Mills field case and the gravitational field studied by Mme Choquet-Bruhat (1964).

Introducing $d/ds = n^i(\partial/\partial x^i)$ we can write (4.13) as

$$\frac{d}{ds} \overset{1}{A}{}' + \sqcup \overset{1}{A}{}' = 0 \tag{4.15}$$

If we introduce now a space-like surface (i.e. transverse to n^i) S with a local parametrisation y, then we can put (at least locally) $x = x(s,y)$ with

$x(0,y) = y$. Then the general solution of (4.15) can be put in the following form:

$$\overset{1}{A}'(s,y;\xi) = T(s,y)\,\theta(y,\xi) \tag{4.16}$$

where $\theta(y,\xi)$ stays for $\overset{1}{A}'(0,y;\xi)$, the initial value of $\overset{1}{A}'$ on the surface S, and T is an operator written symbolically as

$$T(s,y) = \exp\left[-\int_0^s \sqcup(\sigma,y)\,d\sigma\right] \tag{4.17}$$

Notice that notation (4.17) is generally purely symbolic; only in the case when the matrix $\sqcup$ commutes with its derivative with respect to s, the exponential turns out to be just the classical series as for the numbers. However, much can be said about the asymptotic solution just by considering the norm of the operator (4.17); obviously bounded solutions do not exist if the operator (4.17) is not a bounded one.

The result in (4.16) should be expressed in $x = x(s,y)$ and $\xi = \omega\varphi(x)$ only. If we want the solution $A'(x,\xi)$ to be finite for any value of x and ξ, then the operator (4.17) must have the same properties.

Before proceeding further into the investigation of some examples we draw attention to the interpretation of the conditions of order 0, i.e.

$$n^i \overset{1}{A}{}_i^b = 0 \tag{2.13a}$$

as the initial conditions for the system (4.12). Multiplying it by n^k we obtain

$$n^k n^i \partial_i \overset{1}{A}{}_k^{b'} + n^k \sqcup^{bj}_{ck} \overset{1}{A}{}_j^{c'} = 0 \tag{4.18}$$

or, because of the obvious relation,

$$n^i \partial_i n^k = (d/ds)\,n^k = 0 \tag{4.19}$$

$$\frac{d}{ds}(n^k \overset{1}{A})_k^{b'} + n^k \sqcup^{bj}_{ck} \overset{1}{A}{}_j^{c'} = 0 \tag{4.20}$$

This in turn can be regarded as a system of ordinary differential equations for the quantities $n^k \overset{1}{A}{}_k^{b'}$ if the following relation holds:

$$n^k \sqcup^{bj}_{ck} = W_c{}^b n^j \tag{4.21}$$

where $W_c{}^b$ is an arbitrary non-singular matrix.

This is possible if and only if

$$n^k n_j \sqcup^{bj}_{ck} = 0 \tag{4.22}$$

In the radiative coordinates and in the Minkowskian space-time the condition (4.22) reduces just to

$$\sqcup^{b0}_{c0} = 0 \tag{4.23}$$

In such a case the system (4.20) is equivalent to a system of ordinary differential equations (homogeneous) for the quantities $n^k \overset{1}{A}{}_k^{b'}$, and if the condition $n^k \overset{1}{A}{}_k^{b'} = 0$ holds on a hypersurface transverse to n^k, then it holds everywhere in the space-time.

Since we impose the regularity conditions on the functions $\overset{1}{A}{}_k{}^b(x, \xi)$, the same will be true for the expression $n^k \overset{1}{A}{}_k{}^b$.

Now we are ready to proceed with some specific examples.

5. *Radiative Perturbations of Some Trivial Exact Solutions*

Let us now apply this formalism to some known exact solutions of the Yang-Mills field equations. We begin by the most trivial case, when $\overset{0}{A}{}_i{}^b = Q^b A_i(x)$, Q^b being a constant vector in the Lie algebra of the gauge group G. Then all the solutions verify the Maxwell equations *in vacuo*, but do not obey the superposition principle.

Let us suppose that the non-perturbed potential describes a plane wave in the direction of the x-axis:

$$\overset{0}{A}{}_i{}^b = Q^b \overset{0}{A}{}_i(x), \tag{5.1}$$

$$\overset{0}{A}{}_t = 0, \qquad \overset{0}{A}{}_x = 0, \qquad \overset{0}{A}{}_y = \sin(kx - \omega t), \qquad \overset{0}{A}{}_z = \cos(kx - \omega t) \tag{5.2}$$

It is obvious that there exists a solution for $\overset{1}{A}{}_i$ just proportional to $\overset{0}{A}{}_i$, because for such solutions the superposition principle holds (cf. Kerner, 1971). Let us verify then if an asymptotic plane wave solution exists along any other axis which, by a linear transformation, can be chosen as y-axis. So, without loss in generality, we suppose

$$\varphi = k_1 y - \omega_1 t, \qquad \text{with } k_1{}^2 c^2 = \omega_1{}^2 \tag{5.3}$$

Next, introduce the radiative coordinates:

$$x^0 = \varphi = k_1 y - \omega_1 t = u$$
$$s = k_1 y + \omega_1 t,$$

and x, z the same as before. Then

$$\begin{aligned} n_u &= 1, & n_s &= n_x = n_z = 0 \\ n^s &= -2\omega_1{}^2, & n^u &= n^x = n^z = 0 \\ g^{us} &= -2\omega_1{}^2, & g^{ss} &= g^{uu} = 0 \end{aligned} \tag{5.4}$$

We have also

$$\overset{0}{A}{}_u = \overset{0}{A}{}_s = \frac{1}{2k_1} \sin\left(kx - \frac{\omega}{2\omega_1}(s-u)\right)$$

$$\overset{0}{A}{}_x = 0, \qquad \overset{0}{A}{}_z = \cos\left(kx - \frac{\omega}{2\omega_1}(s-u)\right) \tag{5.5}$$

The matrix $\sqcup^{bj}_{ck}$ takes on the following form:

$$-\frac{1}{2\omega_1^2}\sqcup^{bj}_{ck}=(Q^b Q_c - Q^a Q_a \delta_c^b)\, g^{jm} \overset{0}{F}_{sm} \overset{0}{A}_k \tag{5.6}$$

In order to simplify the system (4.12) we have to diagonalise first the matrix $(Q^b Q_c - Q^a Q_a \delta_c^b)$; this can always be done because of its obvious symmetry. Then we obtain the following system for the quantities $\overset{1}{A}{}'_i$: (the prime means derivative with respect to ξ):

$$\frac{d}{ds}\overset{1}{A}{}'_s + 2\omega_1^2\left[\frac{k}{4k_1^2}\cos\lambda\sin\lambda\,\overset{1}{A}{}'_x + \frac{\omega}{4\omega_1 k_1}\sin^2\lambda\,\overset{1}{A}{}'_2\right]=0$$

$$\frac{d}{ds}\overset{1}{A}{}'_x=0 \qquad \frac{d}{ds}\overset{1}{A}{}'_2 + 2\omega_1^2\left[\frac{k}{2k_1}\cos^2\lambda\,\overset{1}{A}{}'_x + \frac{\omega}{2\omega_1}\sin\lambda\cos\lambda\,\overset{1}{A}{}'_2\right]=0$$

$$\frac{d}{ds}\overset{1}{A}{}'_u - 2\omega_1^2\frac{\omega\omega_1}{2k_1^2}\cos\lambda\sin\lambda\,\overset{1}{A}{}'_s + 2\omega_1^2\frac{k}{4k_1^2}\cos\lambda\sin\lambda\,\overset{1}{A}{}'_x$$

$$+\frac{\omega}{4\omega_1 k_1}\sin^2\lambda\,\overset{1}{A}{}'_2=0$$

with

$$\lambda = kx - \frac{\omega}{2\omega_1}(s-u) \tag{5.7}$$

Eliminating $\overset{1}{A}{}'_x$ and $\overset{1}{A}{}'_s$ from these equations by virtue of the necessary (2.13) relation $n^i A'_i = 0$ will give us finally

$$\frac{d}{ds}A'_i = 0 \tag{5.8}$$

i.e. the solution constant along the rays. The complete solution is of the form $P^b A'_i$, where P^b is an eigenvector of the matrix

$$(Q^b Q_c - Q^a Q_a \delta_c^b) \tag{5.8a}$$

As our next example we shall take the following potential:

$$\overset{0}{A}{}_i^b = Q^b\left[\frac{1}{r},0,0,0\right] \tag{5.9}$$

where

$$r^2 = x^2 + y^2 + z^2 \tag{5.10}$$

Let us first look at the spherical asymptotic wave: we take

$$u = \varphi = kr - \omega t, \qquad \text{with } c^2 k^2 = \omega^2$$
$$s = kr + \omega t \tag{5.11}$$

and the angles ϑ, ϕ unchanged. In these coordinates we have

$$\overset{0}{A}_i = \left[-\frac{1}{2\omega(s+u)}, \frac{1}{2\omega(s+u)}, 0, 0\right] \tag{5.12}$$

and the only non-vanishing component of $\overset{0}{F}_{ij}$ is

$$\overset{0}{F}_{us} = -\frac{2k}{(s+u)^2} \tag{5.13}$$

The only non-vanishing components of the matrix $\sqcup_{ck}^{bj}$ are

$$-\sqcup_{cu}^{bs} = \sqcup_{cs}^{bs} = (Q^b Q_c - Q^a Q_a \delta_c{}^b)\frac{8\omega^2 k^2}{(s+u)^3} + \delta_c{}^b \frac{1}{(s+u)} \tag{5.14}$$

As before, from the zero-order condition we have $\overset{1}{A}'_s = 0$, so that finally we obtain the following set of equations (after diagonalising the matrix (5.8a)):

$$\frac{d}{ds}\overset{1}{A}'_i + \frac{1}{(s+u)}\overset{1}{A}'_i = 0 \tag{5.15}$$

and the solutions are the usual spherical waves.

As a third example we take an asymptotic plane wave with a static potential Q^a/r as a background. We put

$$u = \varphi = kx - \omega t, \qquad s = kx + \omega t$$

y, z the same then we get

$$A_u = -A_s = \left[-\frac{1}{2\omega r}, \frac{1}{2\omega r}, 0, 0\right] \tag{5.16}$$

and the non-vanishing components of the tensor F_{ij} are

$$F_{su} = \frac{2k}{\omega r^2}, \qquad F_{yu} = -F_{ys} = \frac{y}{\omega r^3}$$

$$F_{zu} = -F_{zs} = \frac{z}{\omega r^3} \tag{5.17}$$

with

$$r^2 = \frac{(s+u)^2}{4k^2} + y^2 + z^2$$

Then we get the following equations for A'_i (after diagonalisation of the matrix (5.8)):

$$\frac{d}{ds}\overset{1}{A}'_u = 0 \qquad \frac{d}{ds}\overset{1}{A}'_y = \frac{d}{ds}\overset{1}{A}'_z = 0$$

$$\frac{d}{ds}\overset{1}{A}'_s - \frac{y}{4\omega^2 r^5}\overset{1}{A}'_y - \frac{2}{4\omega^2 r^5}\overset{1}{A}'_z = 0 \tag{5.17}$$

Because of $n^i \overset{1}{A}'_i = 0$ we get $\overset{1}{A}'_s = 0$.

The solution again is a plane wave with the subsidiary condition

$$y\overset{1}{A}'_y + z\overset{1}{A}'_z = 0 \tag{5.18}$$

All the aforementioned cases verify, of course, the condition (4.22).

6. *Conclusion*

The few examples we have considered above give us very restrictive results: the only possible asymptotic solutions which can be superposed with the trivial background field by means of the W.K.B. technique are themselves trivial; there is almost no trace of the interaction of the Yang-Mills field with itself. There are two main reasons for this. First, we have looked only upon the trivial background; second, that we were developing the asymptotic wave with the parameter ω^{-1}, which means that the energy of the asymptotic wave is of the same order as the energy of the background field. The results are thus comparable to the solutions obtained by just considering the possibilities of superposition of the factorisable solutions of the electro-magnetic type (cf. Kerner, 1971). Different and more interesting results can be obtained by modifying the W.K.B. method and assuming that the energy of the asymptotic wave is very small as compared to the energy of the background field, and using the ratio of the two energies as a parameter in the development.

These results will be given in our next paper.

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